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$P(4)$ Affine and Superhamiltonian Formulations of Charged Particle Dynamics

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We establish a correspondence between the recently proposed $P(4)$ affine and the standard superhamiltonian descriptions of the electrodynamics of classical charged particles. The $P(4)$ theory uses a generalized affine connection on the affine frame bundle $A(M)$ over spacetime, and an affine connection is induced on phase space thought of as the vector bundle T^*M. On the phase space manifold T^*M this affine structure defines a covariant canonical symplectic form, which, when coupled with the canonical free-particle superhamiltonian, reproduces the Lorentz force law for classical charged particles. Conversely, one may "split" the noncanonical symplectic form on T^*M to define an affine connection on $A(M)$ and thus return to the $P(4)$ theory from symplectic geometry. The correspondence also allows a geometrization of superhamiltonian dynamics. Roughly speaking, the symplectic form on T^*M is geometrized as an R^4-affine connection on $A(M)$, and the superhamiltonian is geometrized as an affine difference function on the local momentum-energy tangent affine spaces.

1. INTRODUCTION

The unified theory of gravitation and electromagnetism suggested recently (Norris, 1985) is a geometric unified theory based on the Poincaré group $P(4)=O(1,3)\otimes R^4$. Its distinctive and most important features are that: (1) it uses a geometry with a generalized affine connection on an affine frame bundle as underlying geometric structure, and (2) it interprets the truly affine part of geometry as freedom in the choice of the origin in the local momentum-energy spaces. The "natural" choice of an origin field allows one to reproduce the Einstein-Maxwell equations in the $P(4)$ theory as geometric equations stated in terms of the $P(4)$ curvature tensor. Physically, the natural choice of the origin field is related to the use of instantaneously comoving observers in describing the motion of charged test

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particles. In the $P(4)$ theory the trajectory of a particle with mass m and affine charge ε is identified with an affine geodesic of type ε, and when $m \neq 0$ this equation is the Lorentz force law for a particle with charge-to-mass ratio ε.

The fundamental geometrical idea in the $P(4)$ theory is to model the momentum-energy of classical charged particles as affine vectors. Intuitively, an affine vector is a vector known only up to a translation, but the "differences" between affine vectors are (unqualified) vectors. Since the canonical momentum-energy of a classical charged particle translates under electromagnetic gauge transformations, it is clear that the affine vector model could apply to the canonical momentum-energy; however, it does not apply to the kinetic momentum-energy, $m\, d\gamma/ds$, which is a well-defined linear vector field along the world line of a particle. It is therefore natural to inquire into the relationship of affine geometry to the symplectic geometry of Hamiltonian dynamics where the canonical momentum-energy plays such a fundamental role. It is our purpose here to clarify this relationship.

In this paper we expand the physical and mathematical foundations of the $P(4)$ theory and relate the affine geometric structure of the theory to the symplectic geometry of Hamiltonian dynamics. Our main result is that we establish a correspondence between these two descriptions of the electrodynamics of classical charged particles. The $P(4)$ theory uses an affine connection on the affine frame bundle $A(M)$ over spacetime, and an affine connection is induced on phase space thought of as the cotangent bundle T^*M. On the phase space manifold T^*M this affine structure defines a covariant canonical symplectic form which, when coupled with the canonical free-particle superhamiltonian, reproduces the Lorentz force law for classical charged particles. Conversely, one may "split" the noncanonical symplectic form on T^*M to define an affine connection on $A(M)$ and thus return to the $P(4)$ theory from symplectic geometry. The correspondence also allows a geometrization of superhamiltonian dynamics. Roughly speaking, the symplectic form on T^*M is geometrized as an R^4-affine connection on $A(M)$, and the superhamiltonian is geometrized as an affine difference function.

We begin in Section 2 with a brief survey of those ideas from differential affine geometry needed for our discussion. In Section 3 we set up an affine vector model of momentum-energy using the standard observational description of classical charged particles in electromagnetic fields. Instantaneously comoving observers are used to define a fundamental affine vector field $\hat{0}$ on spacetime; we refer to this field as the charged classical vacuum because it represents the state of zero kinetic momentum-energy of classical charged particles relative to uncharged particles. The observational description also provides the differential affine transport law for the charged vacuum, and we identify the Maxwell field tensor with the R^4-translational connection relative to $\hat{0}$. We define affine geodesics with respect to such an electromagnetic affine connection and thereby recover the Lorentz force law. R^4-curvature and integrability conditions are introduced, and we show that integrable electromagnetic affine connections correspond to covariant constant electromagnetic fields on flat spacetime.

In order to establish a relationship between affine momentum-energy as defined in Section 3 with canonical momentum-energy, we turn in Section 4 to the symplectic geometry of Hamiltonian mechanics. We consider two alternative ways of including the electromagnetic interaction in superhamiltonian dynamics. Starting from the free-particle system (free-particle superhamiltonian and canonical symplectic form), one can modify either the superhamiltonian or the symplectic form to introduce the electromagnetic interaction. The resulting formulations in fact represent the same physical system, but expressed with respect to different phase space coordinates. The transformation between the systems is the well-known substitution rule $(x^\mu, \pi_\nu) \to (x^\mu, \pi_\nu - eA_\nu)$, which is an affine translation of momentum-energy. This establishes an R^4 affine translational symmetry for the superhamiltonian dynamical description of classical charged particles in electromagnetic fields.

In Sections 5 and 6 we establish the correspondence between the $P(4)$ and superhamiltonian descriptions. We rewrite the dynamical Hamiltonian equation of motion in Section 5 as a "semihamiltonian" equation, and make the assumption that the spacetime geometrical model of canonical momentum-energy is the affine vector model. In order to tie down a correspondence with the $P(4)$ theory, we need to relate a choice of coordinates on T^*M to a choice of "origin" of an affine frame field on M. We show that the identification of the charged vacuum $\hat{0}$ with coordinates in which the superhamiltonian is the free particle superhamiltonian reduces the semihamiltonian equation of motion to the affine geodesic equation.

In Section 6 we complete the affine reinterpretation of superhamiltonian dynamics. By studying the change in form of the Hamilton equations for free uncharged particles under a redefinition of the reference momentum-energy, modeled geometrically as momentum-energy translations, we identify an additive integrable part of the electromagnetic R^4 affine connection measured relative to the field $\hat{0}$. We show that the noncanonical symplectic form on T^*M can be rewritten in a covariant canonical form, and that this 2-form can be split to obtain the vector bundle definition of an affine connection. This leads us back to the $P(4)$ theory.

We then show that Hamilton's canonical equations can be replaced by a pair of affine-geometric equations. The Hamilton equation that defines the 4-velocity of a particle is replaced by a geometric definition in terms of the affine difference function. The dynamical Hamiltonian equation is then replaced by the affine geodesic equation. Section 7 contains a discussion of our results and concluding remarks. The general correspondence between the $P(4)$ and superhamiltonian formulations is displayed in Table II in Section 7.

2. AFFINE GEOMETRY

In the sections that follow we describe the $P(4)$ picture of the local momentum-energy spaces $\Pi_p M$ as 4-dimensional affine spaces, and in order to have a standard notation, we present the following brief survey of affine geometry.

Formally (Dodson and Poston, 1977) an *affine space* is a triple (S, V, δ), where S is a set whose elements are called *points*, or *affine vectors*, V is an n-dimensional linear vector space, and $\delta : S \times S \to V$ is the *difference function*, with the following properties:

(i) $\delta(x, y) + \delta(y, z) = \delta(x, z) \quad \forall x, y, z \in S.$
(ii) for each $x \in S$, the map $\delta_x : S \to V$ defined by

$$\delta_z(y) := \delta(y, x) \tag{1}$$

is a bijection.

It is often convenient to be less precise and refer to the set S as the affine space.

The fundamental geometrical operation in an affine space is the operation of taking the difference between points using δ, and this difference is represented by a vector. A well-known example is Minkowski spacetime M_0 together with the atlas of all $P(4)$-related Lorentz charts (M_0, x^μ). In Minkowski spacetime all tangent spaces coincide. Then, for any fixed point $p_0 \in M_0$, $(M_0, T_{p_0}M_0, \delta_0)$ is an affine space with difference function

$$\delta_0(p, q) = [x^\mu(p) - x^\mu(q)]r_\mu \tag{2}$$

Here (r_μ) denotes the standard basis of R^4. This affine structure considers the fundamental points as the events of flat spacetime. On the other hand, in the $P(4)$ theory the fundamental points are considered to be the momentum-energies of particles.

Property (i) is the statement that the differences between points in an affine space obey a triangle equality. It implies, in particular, that $\delta(x, x) = 0$ for all $x \in S$. That is, the difference between a "point and itself" is zero (the zero vector!). Property (ii) states that the affine space S can be made isomorphic to its vector space V simply by choosing any point, say x, to identify with the zero vector in V, since $\delta_x(x) = \delta(x, x) = 0$.

An *affine frame* for (S, V, δ) is a pair (e_μ, x), where (e_μ) is a linear frame in V and x is a point in S, the *origin* of the affine frame. The "coordinates" of a point $y \in S$ relative to an affine frame (e_μ, x) can be split into two parts:

(i) ${}^x y := \delta_x(y) =$ the radius vector of y relative to the origin x.

(ii) $({}^x y)^\mu := e^\mu({}^x y) =$ the linear components of y relative to the affine frame.

Here (e^μ) denotes the coframe dual to (e_μ). Note that "radius" vector can be defined relative to any fixed point in S without having to specify a complete affine frame.

The affine group $A(4) = Gl(4) \otimes R^4$ acts on affine frames in the following way. For $a = (a^\nu_\mu) \in Gl(4)$ and $\eta = (\eta^\mu) \in R^4$ the $A(4)$ element (a, η) transforms the affine frame (e_μ, x) into the affine frame $(\bar{e}_\mu, \bar{x}) = (e_\nu a^\nu_\mu, x + \eta^\mu e_\mu)$. The linear frame thus undergoes a $Gl(4)$ linear transformation, while the origin of the frame undergoes a translation. When the linear frames under consideration are orthonormal frames relative to a spacetime metric, then the Poincaré subgroup $P(4) \subset A(4)$ is the appropriate transformation group. In this paper we are mainly concerned with the component of the $P(4)$ connection that models the electromagnetic field, namely the R^4 component. For electrodynamics in flat spacetime the $P(4)$ theory should be thought of as an R^4 theory.

The simplest possible affine space is obtained by giving an n-dimensional vector space its *natural affine structure.* This means to consider the triple $(\hat{V}, V, \delta_0)$, where $\hat{V} = V$ as sets, and the difference function δ_0 is subtraction as defined in V. This simple example means that intuitively we may think of an affine space as a vector space in which the zero vector is put on equal footing with all other vectors in the space. We denote by A^4 the affine space obtained by giving R^4 its natural affine structure.

The affine structure assumed in the $P(4)$ theory is $(\Pi_p M, T_p M, \delta)$ at each spacetime point $p \in M$, where $\Pi_p M$ denotes the space of momentum-energies at p. The difference function δ is the new geometrical object, and as we will see, it is fundamentally related to the superhamiltonian function on phase space. When necessary we will denote affine vectors in $\Pi_p M$ as characters with "hats." Thus, if $\hat{\pi}_1$ and $\hat{\pi}_2$ are momentum-energy affine vectors in $\Pi_p M$, then their difference is a vector v defined by

$$v := \delta(\hat{\pi}_1, \hat{\pi}_2) \tag{3}$$

If we keep in mind that affine vectors are not linear vectors, then a convenient shorthand notation for this difference is

$$\hat{\pi}_2 = \hat{\pi}_1 \oplus v \tag{4}$$

Given an affine vector $\hat{\pi}_1$ and a linear vector v, one may define $\hat{\pi}_2$ by equation (4). It is important to note that in so doing the difference function has been used in an essential way. Specifically, $\hat{\pi}_2$ is determined as $\delta^{-1}_{\hat{\pi}_1}(v)$.

We next extend these algebraic ideas from a spacetime event p to a 4-dimensional spacetime region. An *affine frame field* f is a pair $f=(e_\mu, \hat{\pi})$, where (e_μ) is a linear frame field and $\hat{\pi}$ is an *origin* (affine vector) *field*. A *generalized affine connection* ω may be specified (Kobayashi and Nomizu, 1963) by a pair $\omega=(\Gamma, \hat{K})$, where Γ is a linear connection and $\hat{K}$ is an R^4 translational connection. The components $({}^f\Gamma, {}^fK)$ of ω relative to an affine frame field $(e_\mu, \hat{\pi})$ can be defined by the exterior covariant derivative formulas (Cartan's structure equations)

$$\hat{D}e_\mu = De_\mu = {}^f\Gamma^\nu_\mu e_\nu \tag{5}$$

$$\hat{D}\hat{\pi} = {}^fK^\mu e_\mu \tag{6}$$

In these equations ${}^f\Gamma^\nu_\mu$ are the component 1-forms of a linear connection, and ${}^fK^\mu$ is the R^4-valued 1-form component of the translational part of the connection. The notation is that $\hat{D}$ denotes exterior covariant differentiation with respect to ω, while D denotes exterior covariant differentiation (for linear geometric objects) with respect to Γ. Equation (5) implies $\hat{D}T=DT$ for any linear tensor T.

The transformation properties of Γ are well known and will not be discussed. We remark only that under a change of affine frame $f\to f'$ the components of ${}^{f'}\Gamma$ depend on ${}^f\Gamma$, but not on ${}^fK$.

Let us consider only affine frame transformations of the type $f=(e_\mu, \hat{\pi})\to(e_\mu, \hat{\sigma})$. We may then use in place of equation (4) the more explicit notation

$$\hat{D}\hat{\pi} = {}^{\hat{\pi}}K \tag{7}$$

For $f'=(e_\mu, \hat{\sigma})$ we also have the equation

$$\hat{D}\hat{\sigma} = {}^{\hat{\sigma}}K \tag{8}$$

Let t be the vector field such that $\hat{\pi}=\hat{\sigma}\oplus t$. We assume

$$\hat{D}(\hat{\sigma}\oplus t) := \hat{D}\hat{\sigma} + Dt \tag{9}$$

Equations (5)-(8) can now be used to show that

$${}^{\hat{\pi}}K = {}^{\hat{\sigma}}K + Dt \tag{10}$$

This is the transformation law for $\hat{K}$ under change of origin field.

The curvature tensor Ω of the affine connection $\omega=(\Gamma, \hat{K})$ splits into a linear part Ω_L and a translational part $\hat{\Omega}_T$, and may be defined by computing the second exterior covariant derivatives $\hat{D}^2e_\mu = D^2e_\mu$ and

$\hat{D}^2\hat{\pi} = D(\hat{D}\hat{\pi})$ using (5) and (6). Assuming that the linear connection is torsion-free, one finds

$$^f\Omega_L = d^f\omega + {}^f\omega \wedge {}^f\omega \tag{11}$$

$$^f\Omega_T = d^fK + {}^f\omega \wedge {}^fK \tag{12}$$

Although the difference function is a basic feature in affine geometry, it does not correspond to a tensor field on spacetime and thus one has less intuition about its physical interpretation. After studying superhamiltonian dynamics we will be able to make some remarks concerning the interpretation of δ as a "generalized" superhamiltonian. For the most part we will operate formally with δ, assuming that it is smooth in some appropriate sense. We view equation (9) as a "compatibility" condition between the R^4 connection and the difference function. Recalling equations (3) and (4), we can express the compatibility condition (9) as

$$D(\delta(\hat{\pi}, \hat{\sigma})) = \delta_0(\hat{D}\hat{\pi}, \hat{D}\hat{\sigma}) \tag{13}$$

3. AFFINE MOMENTUM–ENERGY

Consider a classical charged particle moving along a trajectory $\gamma(s)$ in an electromagnetic field $F = F^{\mu}_{\nu} e_{\mu} \otimes e^{\nu}$ in flat spacetime. Since such a particle obeys the Lorentz force law, the trajectory may be determined from initial conditions and knowledge of the Maxwell field tensor. Observationally this means: "Actual world line compared to world line of fiducial test particle passing through same point with same 4-velocity" (Misner *et al.*, 1973). That is, at each point along the trajectory introduce as a standard of reference an instantaneously comoving observer that is uncharged and freely falling. We are going to show that this pointwise definition, when "glued" together over a 4-dimensional spacetime region, leads to an affine structure for the local momentum–energy spaces, and that it also provides a differential transport law for the zero of momentum–energy of classical charged particles relative to uncharged particles.

Consider a spacetime region U with electromagnetic field tensor F, and a collection of classical charged test particles on U. Since an instantaneously comoving uncharged test particle cannot distinguish between charged test particles with the same charge-to-mass ratio, we introduce a parameter $\varepsilon = q/m$, and write the Lorentz force law as

$$d\dot{\gamma}/ds = \varepsilon F(\dot{\gamma}) \tag{14}$$

Along the trajectory of the particle the kinetic momentum–energy of the classical charged particle (per unit rest mass) satisfies

$$u = \dot{\gamma} \tag{15a}$$

$$d_{\dot{\gamma}} u = \varepsilon F(\dot{\gamma}) \tag{15b}$$

Here $d_{\dot{\gamma}}u$ denotes the directional derivative of u in the direction of $\dot{\gamma}$. As remarked earlier, the kinetic momentum-energy is a linear vector field along the trajectory.

At each point $\gamma(s)$ the instantaneously comoving observer may be defined by

$$p_0 = \dot{\gamma} \tag{16a}$$

$$d_{\dot{\gamma}} p_0 = 0 \tag{16b}$$

Equations (16a) and (16b) define an instantaneous comoving reference of momentum-energy (per unit rest mass) pointwise along the trajectory of the classical charged particle. Equation (16b) is the transport law for this comoving reference momentum-energy, defined at the moment only along one trajectory.

At each spacetime event u and p_0 agree, and we express this pointwise relationship as

$$u = p_0 - 0 \tag{17}$$

We assume [cf. equation (24)]

$$d_{\dot{\gamma}}u = d_{\dot{\gamma}} p_0 = d_{\dot{\gamma}}0 \tag{18}$$

Using equation (15b), we derive

$$\varepsilon F(\dot{\gamma}) = d_{\dot{\gamma}} p_0 - d_{\dot{\gamma}}0 \tag{19}$$

Temporarily (and incorrectly!) we rewrite this equation with the help of equation (16b) as

$$d_{\dot{\gamma}}0 = -\varepsilon F(\dot{\gamma}) \tag{20}$$

We will make the necessary correction below.

We imagine that the process is repeated for all possible classical charged particles passing through each event of $\gamma(s)$, and then extend the procedure to all points of U. Then equation (20) can be written as

$$d_t 0 = -\varepsilon F(t) \tag{21}$$

valid for all *timelike* vectors t at each point of U. If we assume that equation (21) is true for all vectors at each spacetime event, then it may be generalized to

$$d0 = -\varepsilon F \tag{22}$$

This equation is now the differential transport law for the zero momentum-energy field for classical charged particles relative to instantaneously comoving uncharged test particles.

Equation (22) may appear rather puzzling. How can the derivative of zero not be zero! The reason of course is that the object that we have written as 0 in the equation is not the zero vector field. Rather, it represents the state of zero momentum-energy for classical charged particles defined relative to instantaneously comoving uncharged particles, and identified with the zero vector at each spacetime event. It does *not* follow that it has the geometrical representation as the zero vector field. From equation (17) we see that once a specific charged particle is specified at a spacetime event, it is only the *difference* between p_0 and 0 that is a linear vector field $\dot{\gamma}(s)$ along the world line of the particle. We make the assumption that p_0 and 0 are affine vectors $\hat{p}_0$ and $\hat{0}$, and that the geometrical meaning of equation (17) is

$$\dot{\gamma}=\delta(\hat{p}_0,\hat{0}) \tag{23}$$

Let $\hat{D}$ denote an affine covariant derivative with linear part D. Applying formula (9) to equation (23), we obtain

$$D_{\dot{\gamma}}\dot{\gamma}=D_{\dot{\gamma}}\delta(\hat{p}_0,\hat{0})=\hat{D}_{\dot{\gamma}}\hat{p}_0-\hat{D}_{\dot{\gamma}}\hat{0} \tag{24}$$

We interpret the comoving condition to mean $\hat{D}_{\dot{\gamma}}\hat{p}_0=0$, so that equation (24) reduces with the help of equation (15b) to

$$\hat{D}_{\dot{\gamma}}\hat{0}=-D_{\dot{\gamma}}=-\varepsilon F(\dot{\gamma}) \tag{25}$$

The affine interpretation of equation (22) is thus

$$\hat{D}\hat{0}=-\varepsilon F \tag{26}$$

According to equation (7), we can make the identification

$$^{\hat{0}}K=-\varepsilon F \tag{27}$$

The Maxwell field tensor is proportional to the R^4 connection relative to the comoving momentum-energy affine vector field $\hat{0}$. Equation (27) is both geometrical and physical. It is the geometrical model of the observational fact that the instantaneous change in momentum-energy of charged particles in electromagnetic fields, measured relative to uncharged inertial observers, is described by the Lorentz force law. Geometrically, it models the Maxwell field tensor as the component of an R^4 affine connection relative to the affine field $\hat{0}$.

In the procedure we followed above we started with the equation of motion for a classical charged particle and the defining properties of the comoving observer, and we inferred the existence of the affine vector field $\hat{0}$. Now, although u and p_0 are only defined along curves, $\hat{0}$ is defined everywhere and is independent of the specific choice of particle. We therefore consider $\hat{0}$ as a fundamental field defined on spacetime, and because it represents the state of zero momentum–energy of charged particles relative to uncharged particles, we will refer to $\hat{0}$ as the field of the *charged classical vacuum.*

Consider Minkowski spacetime with a charged vacuum field $\hat{0}$ satisfying the transport law (26). Let $(\gamma(s), m, \varepsilon)$ denote an arbitrary charged particle (Norris, 1985), where we do not assume that $\gamma(s)$ is a solution of equation (14).

Definition. The momentum–energy per unit rest mass of a classical charged particle $(\gamma(s), m, \varepsilon)$ is the affine vector field $\hat{p}$ defined along $\gamma(s)$ by

$$\hat{p} = \hat{0}(\gamma(s)) \oplus \dot{\gamma}(s) \tag{28}$$

Using equations (9) and (26), one can express the affine covariant derivative of $\hat{p}$ along $\gamma(s)$ as

$$\hat{D}_{\dot{\gamma}}\hat{p} = \hat{D}_{\dot{\gamma}}\hat{0} + d_{\dot{\gamma}}\dot{\gamma} = -\varepsilon F(\dot{\gamma}) + d_{\dot{\gamma}}\dot{\gamma} \tag{29}$$

If $\hat{p}$ is constant along γ with respect to the affine connection, then $\hat{D}_{\dot{\gamma}}\hat{p} = 0$, and equation (29) reduces to the Lorentz force law (14) for a particle with charge-to-mass ratio ε. Such a curve may be considered as an *affine geodesic* of type ε (Norris, 1985). The physical interpretation is that a Lorentz force law trajectory $\gamma(s)$ is one along which the momentum–energy $\hat{p}$, obtained by translating the charged vacuum by $\dot{\gamma}$ as in (28), is constant (affine parallel) along the trajectory. A charged particle will seek out a world line along which the variation $\hat{D}_{\dot{\gamma}}\hat{0}$ in the charged vacuum is just balanced by the linear change $d_{\dot{\gamma}}\dot{\gamma}$ in the kinetic momentum–energy per unit rest mass.

By equations (12) and (27), the R^4 curvature relative to the charged classical vacuum is

$${}^{\hat{0}}\Omega_T = -\nabla_{[\mu}F^{\lambda}_{\nu]}\partial_\lambda \otimes dx^\mu \wedge dx^\nu \tag{30}$$

Let $\Phi^{\lambda}_{\mu\nu}$ denote the components of ${}^{\hat{0}}\Omega_T$ so that

$$\Phi^{\lambda}_{\mu\nu} = -\nabla_{[\mu}F^{\lambda}_{\nu]} \tag{31}$$

The transport law (26) will be *integrable* when $\Phi = 0$. In this case we can choose a zero at one spacetime event and then use the transport law (26) to define zero throughout a neighborhood, and the result will be path-independent.

In this section we have assumed from the beginning that the tensor field F was an electromagnetic field tensor. However, we have not used the Maxwell equations anywhere in the discussion, and provided $F_{\mu\nu} = -F_{\nu\mu}$, the general features of the discussion remain valid if F is an otherwise arbitrary tensor field. The integrability condition $\Phi^{\lambda}_{\mu\nu}=0$ implies that $\Phi^{\mu}_{\mu\nu}=0$ and $\Phi_{[\mu\nu\lambda]}=0$ when Φ vanishes. Using (31), it is not difficult to show that

$$\Phi^{\mu}_{\mu\nu}=0 \Rightarrow \nabla_{\mu}F^{\mu}_{\nu}=0 \tag{32}$$

$$\Phi_{[\mu\nu\lambda]}=0 \Rightarrow \nabla_{[\mu}F_{\nu\lambda]}=0 \tag{33}$$

The result is that if an R^4 connection F is antisymmetric and the R^4 curvature is integrable, then F must satisfy the source-free Maxwell equations. Combining equations (32) and (33) with $\Phi^{\lambda}_{\mu\nu}=0$, one finds that in fact F must be covariant constant:

$$\nabla_{\mu}F_{\nu\lambda}=0 \tag{34}$$

The equations $\Phi^{\mu}_{\mu\nu}=0$ and $\Phi_{[\mu\nu\lambda]}=0$ were adopted as the R^4 electromagnetic field equations without the stronger condition (34), which only holds when $\Phi^{\lambda}_{\mu\nu}=0$.

4. SUPERHAMILTONIAN DYNAMICS OF A CHARGED TEST PARTICLE. TWO ALTERNATIVE DESCRIPTIONS

Let us consider how the geometric structure of the electromagnetic field is related to the dynamical structure involved in the description of particle mechanics. We are going to use superhamiltonian dynamics to ensure that the picture is relativistic, but we will not make all the constructions generally relativistic. More specifically, we are going to use Minkowski metric

$$\eta_{\mu\nu}=\mathrm{diag}(-1,1,1,1) \tag{35}$$

rather than a general metric $g_{\mu\nu}$.

There are two ways to formulate the suerphamiltonian dynamics of a charged particle placed in an electromagnetic field. Both of them are obtained by modifying the superhamiltonian dynamics of a neutral (uncharged) particle. The case of a neutral particle is trivial, but we include its description here to provide a basis for comparison and later discussion, where we demonstrate the difference between the standard and the $P(4)$ theory approaches.

The superhamiltonian dynamics of a particle in the general case can be described as follows (Godbillon, 1969). Let T^*M be the cotangent bundle of spacetime M. Then the dynamics of the particle is defined by two elements: a symplectic form S on T^*M (a closed 2-form of maximal rank) and superhamiltonian $\mathscr{H}$ on T^*M (a real-valued function on T^*M). If $X: T^*M \to T(T^*M)$ is a vector field on T^*M, then it is called a superhamiltonian dynamic system if and only if it satisfies the equation

$$d\mathscr{H}=-S \lrcorner X \tag{36}$$

A curve $f:(a, b) \to T^*M$ $[\lambda \mapsto f(\lambda)$, where $\lambda \in (a, b)$ and $f(\lambda) \in T^*M]$ is called an integral curve of the superhamiltonian system if

$$df(d/d\lambda)_\lambda = f_*(d/d\lambda)_\lambda = X(f(\lambda)) \tag{37}$$

where X is a superhamiltonian dynamic system and (a, b) is an open interval of the real axis. Any integral curve of the superhamiltonian system satisfies the Hamilton equations, which can be written in general coordinate-free form as

$$d\mathcal{H} = -S \lrcorner f_*(d/d\lambda) \tag{38}$$

In local coordinates (x^μ, p_ν) of T^*M a general vector field can be written as

$$X = a^\mu \frac{\partial}{\partial x^\mu} + b_\nu \frac{\partial}{\partial p_\nu} \tag{39}$$

and the superhamiltonian $\mathcal{H}$ can be considered as a function of x^μ, p_ν:

$$\mathcal{H} = \mathcal{H}(x^\mu, p_\nu) \tag{40}$$

Then

$$d\mathcal{H} = \frac{\partial \mathcal{H}}{\partial x^\mu} dx^\mu + \frac{\partial \mathcal{H}}{\partial p_\nu} dp_\nu \tag{41}$$

The symplectic form S can be represented as

$$S = S_{ik} dy^i \wedge dy^k, \qquad i, k = 1, \ldots, 2n \tag{42}$$

where $y^1 = x^1, \ldots, y^n = x^n$, $y^{n+1} = p_1, \ldots, y^{2n} = p_n$, and S_{ik} is an antisymmetric matrix of rank $2n$. The matrix (S_{ik}) can be represented in the general case as

$$(S_{ik}) = \begin{pmatrix} C & B \\ -B & D \end{pmatrix} \tag{43}$$

where B is an $(n \times n)$ matrix and C, D are $(n \times n)$ antisymmetric matrices. In all the situations considered below we use only the symplectic forms for which $B = -I$, where I is the $(n \times n)$ unit matrix, and $D = 0$. Thus, the most general form of the matrix S_{ik} used below will be

$$(S_{ik}) = \begin{pmatrix} C & -I \\ I & O \end{pmatrix} \tag{44}$$

i.e., if $C = (C_{\mu\nu})$, then the local expression for the symplectic form S will be

$$S = dp_\mu \wedge dx^\mu + C_{\mu\nu} dx^\mu \wedge dx^\nu \tag{45}$$

A curve f in $T^*(M)$ can be locally expressed by $2n$ functions

$$\lambda \mapsto x^\mu(\lambda); \qquad \lambda \mapsto p_\nu(\lambda) \tag{46}$$

The local expression of the tangent vector field of the curve is then

$$f_*\left(\frac{d}{d\lambda}\right) = \dot{x}^\mu \frac{\partial}{\partial x^\mu} + \dot{p}_\nu \frac{\partial}{\partial p_\nu} \tag{47}$$

where $\dot{x}^\mu = dx^\mu/d\lambda$, $\dot{p}_\nu = dp_\nu/d\lambda$.

Thus, the Hamiltonian equation (38) for the integral curves can be expressed locally as

$$\frac{\partial \mathcal{H}}{\partial x^\mu} dx^\mu + \frac{\partial \mathcal{H}}{\partial p_\nu} dp_\nu = (dp_\mu \wedge dx^\mu + C_{\mu\nu} dx^\mu \wedge dx^\nu) \lrcorner \left(\dot{x}^\mu \frac{\partial}{\partial x^\mu} + \dot{p}_\nu \frac{\partial}{\partial p_\nu} \right) \tag{48}$$

After using the relations $dx^\mu \lrcorner \partial/\partial x^\nu = \delta^\mu_\nu$, $dp_\mu \lrcorner \partial/\partial p_\nu = \delta^\nu_\mu$, and $dx^\mu \lrcorner \partial/\partial p_\nu = dp_\mu \lrcorner \partial/\partial x^\nu = 0$, we find that equation (48) reduces to

$$\frac{\partial \mathcal{H}}{\partial x^\mu} dx^\mu + \frac{\partial \mathcal{H}}{\partial p_\nu} dp_\nu = \dot{x}^\nu \, dp_\nu - (\dot{p}_\mu - C_{\mu\alpha} \dot{x}^\alpha) \, dx^\mu \tag{49}$$

Equation (49) thus implies the Hamiltonian equations

$$\dot{x}^\nu = \partial \mathcal{H} / \partial p_\nu \tag{50}$$

$$\dot{p}_\mu = -\partial \mathcal{H} / \partial x^\mu + C_{\mu\alpha} \dot{x}^\alpha \tag{51}$$

The superhamiltonian dynamics of a free (neutral) particle corresponds to the choice of the symplectic form being the canonical symplectic form of the cotangent bundle T^*M,

$$S_c = dp_\mu \wedge dx^\mu \tag{52}$$

i.e., making $C_{\mu\alpha} = 0$, and the choice of superhamiltonian (Misner *et al.*, 1973)

$$\mathcal{H} = \frac{1}{2m} [m^2 + p^\alpha p_\alpha] \tag{53}$$

The Hamilton equations (50), (51) now reduce to

$$\dot{x}^\nu = p^\nu / m \tag{54}$$

$$\dot{p}_\mu = 0 \tag{55}$$

and the equation of motion takes the form

$$(m\dot{x}^\nu)^{\cdot} = 0 \tag{56}$$

Equation (56) could be put in the simpler form

$$\dot{x}^\mu = p^\mu \tag{57}$$

if we replace the parameter λ by $m\lambda$. At present we want only to stress that the momentum-energy of the particle in this case is a linear vector. No affine structure shows up.

The superhamiltonian dynamics of a charged particle in an electromagnetic field can be obtained by modifying the free-particle dynamics in two possible ways. The first possibility is to keep $\mathcal{H}$ unchanged and to replace the canonical symplectic form (52) with the noncanonical symplectic form (Guillemin and Sternberg, 1984)

$$S = S_c + qF \tag{58}$$

where q is the charge of the particle and F is the electromagnetic 2-form. Locally (58) can be expressed as

$$S = dp_\mu \wedge dx^\mu + qF_{\mu\nu}\, dx^\mu \wedge dx^\nu \tag{59}$$

It is easy to see that the symplectic form (59) together with the superhamiltonian of the free particle (53) produces the dynamics of a charged particle placed in an electromagnetic field F. In fact, the Hamilton equations (50) and (51) now take the form ($C_{\mu\nu} = qF_{\mu\nu}$)

$$\dot{x}^\mu = p^\mu / m \tag{60}$$

$$\dot{p}_\nu = eF_{\nu\alpha}\dot{x}^\alpha \tag{61}$$

This gives the standard Lorentz force law equation of motion

$$(m\dot{x}^\nu)^{\cdot} = eF^\nu_{\ \alpha}\dot{x}^\alpha \tag{62}$$

The relation between the momentum-energy of the particle and the velocity of the particle remains the same as in the case of the free particle. However, the symplectic form (58), (59) is no longer canonical.

It is possible to return to the canonical symplectic form just be changing coordinates on the cotangent bundle. In the case of the single-particle dynamics the coordinate transformations are restricted to

$$X^\mu = X^\mu(x^\alpha), \qquad P_\nu = P_\nu(x^\alpha, p_\beta) \tag{63}$$

if we want to interpret observations in spacetime. Simple but lengthy analysis shows that in this case we can restrict ourselves to the transformations (Boisseau and Barrabès, 1979)

$$X^\mu = x^\mu, \qquad P_\nu = p_\nu + f_\nu(x^\alpha) \tag{64}$$

without sacrificing generality.

The last equation implies

$$p_\nu = P_\nu - f_\nu(x^\alpha) \tag{65}$$

$$dp_\nu = dP_\nu - df_\nu \tag{66}$$

Substitution of equation (66) into equation (59) then gives

$$
\begin{aligned}
S &= (dP_\nu - df_\nu) \wedge dx^\nu + qF_{\mu\nu}\, dx^\mu \wedge dx^\nu \\
&= dP_\nu \wedge dx^\nu + (qF_{\mu\nu}\, dx^\mu - df_\nu) \wedge dx^\nu \\
&= dP_\nu \wedge dx^\nu + qF_{\mu\nu}\, dx^\mu \wedge dx^\nu - f_{\nu,\mu}\, dx^\mu \wedge dx^\nu
\end{aligned} \tag{67}
$$

It is obvious that the choice

$$f_\nu = eA_\nu \tag{68}$$

reduces equation (67) to

$$S = dP_\nu \wedge dx^\nu \tag{69}$$

Thus, the symplectic form becomes canonical in variables (x^α, P_μ), where

$$P_\mu = p_\mu + qA_\mu(x^\alpha) \tag{70}$$

Taking into account that $f_\nu(x^\alpha)$, $A_\nu(x^\alpha)$ are functions of the coordinates (x^α) only, one can interpret equations (65) and (70) as *translations* of momentum-energy spaces over the spacetime points (x^α). Then $f_\nu(x^\alpha)$ in (65) and $qA_\nu(x^\alpha)$ in (70), when considered as functions on spacetime, define fields of translations of infinitesimal momentum-energy spaces. Hence, the field of translations $qA_\nu(x^\alpha)$ reduces the symplectic form (59) to the canonical form (69).

The superhamiltonian in new variables (x^α, P_β) can be obtained from (53) by using the following expression for p_ν obtained from (70):

$$p_\nu = P_\nu - qA_\nu \tag{71}$$

The result is (Misner *et al.*, 1973)

$$\mathcal{H}(x^\mu, P_\nu) = \frac{1}{2m}[m^2 + (P^\mu - qA^\mu)(P_\mu - qA_\mu)] \tag{72}$$

The canonical symplectic form (69) and the superhamiltonian in new variables (72) provide the second alternative way to determine the superhamiltonian dynamics of a charged particle in an electromagnetic field.

It is easy to see that

$$\frac{\partial \mathcal{H}}{\partial x^\nu} = -qA_{\alpha,\nu} \frac{P^\alpha - qA^\alpha}{m} \tag{73}$$

$$\frac{\partial \mathcal{H}}{\partial P_\nu} = \frac{P^\nu - qA^\nu}{m} \tag{74}$$

Taking into account that in the new variables the symplectic form is canonical [i.e., $C_{\mu\alpha}=0$ in the Hamilton equations (50) and (51)] and using expressions (73) and (74), we arrive at the Hamilton equations

$$\dot{x}^{\mu}=\frac{P^{\mu}-qA^{\mu}}{m} \tag{75}$$

$$\dot{P}_{\nu}=qA_{\alpha,\nu}\frac{P^{\alpha}+qA^{\alpha}}{m} \tag{76}$$

Equations (75) and (76) give

$$(m\dot{x}_{\nu}+qA_{\nu})^{\cdot}=qA_{\alpha,\nu}\dot{x}^{\alpha} \tag{77}$$

or

$$(m\dot{x}_{\nu})^{\cdot}=q(A_{\alpha,\nu}-A_{\nu,\alpha})\dot{x}^{\alpha} \tag{78}$$

Since

$$F_{\nu\alpha}=A_{\alpha,\nu}-A_{\nu,\alpha} \tag{79}$$

we obtain the Lorentz force equation

$$(m\dot{x}_{\nu})^{\cdot}=qF_{\nu\alpha}\dot{x}^{\alpha} \tag{80}$$

Recall now that $qA^{\mu}(x^{\alpha})$ can be interpreted as the field of translations of infinitesimal momentum-energy spaces. This circumstance suggests the idea that the spacetime model of canonical momentum-energy P_{α} has natural expression in terms of affine structure rather than linear. The geometric implications of this idea will be considered in the next sections. For future purposes it is convenient to rewrite equation (76) or (77) (the first is a truly Hamiltonian equation, while the second is a Lagrangian equation)in 'semihamiltonian" form

$$\dot{P}_{\nu}=qA_{\alpha,\nu}\dot{x}^{\alpha} \tag{81}$$

5. DYNAMICS OF CHARGED PARTICLES AND AFFINE GEOMETRY

To compare the results of both formalisms, we will rewrite the "semihamiltonian" equation (81) in terms of the notation used in Section 3. Thus, we introduce the notation

$$\pi_{\mu}=P_{\mu}/m \tag{82}$$

$$\varepsilon=q/m \tag{83}$$

Using these variables, we reduce (81) to the "semihamiltonian" equation

$$\dot{\pi}_{\nu}=\varepsilon A_{\alpha,\nu}\dot{x}^{\alpha} \tag{84}$$

Rewriting equation (84) in contravariant form, we obtain

$$\dot{\pi}^{\nu} = \varepsilon A_{\alpha}^{;\nu} \dot{x}^{\alpha} = \varepsilon \eta^{\nu\beta} A_{\alpha,\beta} \dot{x}^{\alpha} \tag{85}$$

As in Section 3, we relate $\dot{\gamma}$ with x^{α} by means of

$$\dot{\gamma} = \dot{x}^{\alpha}\, \partial/\partial x^{\alpha} \tag{86}$$

Here $\dot{\gamma}$ is a linear vector at each point of the particle world line (4-velocity vector). We further introduce the notation

$$d_{\dot{\gamma}} \vec{\pi} = \dot{\vec{\pi}} = \dot{\pi}^{\nu} \frac{\partial}{\partial x^{\nu}} \tag{87}$$

$$\overline{d\vec{A}} = A_{\alpha}^{;\nu} \frac{\partial}{\partial x^{\nu}} \otimes dx^{\alpha} \tag{88}$$

$$\vec{A} = A^{\mu} \frac{\partial}{\partial x^{\mu}} \tag{89}$$

Notice here the difference between $\overline{d\vec{A}}$ and $d\vec{A}$. The former is defined by equation (88), while the latter means

$$d\vec{A} = A^{\alpha}_{,\nu} \frac{\partial}{\partial x^{\alpha}} \otimes dx^{\nu} \tag{90}$$

It is clear, then, that

$$\overline{d\vec{A}} \lrcorner \dot{\gamma} = A_{\alpha}^{;\nu} \dot{x}^{\alpha} \frac{\partial}{\partial x^{\nu}} \tag{91}$$

$$d\vec{A} \lrcorner \dot{\gamma} = A^{\nu}_{,\alpha} \dot{x}^{\alpha} \frac{\partial}{\partial x^{\nu}} \tag{92}$$

We also stress here that the notation F in Section 3 was used for the vector-valued 1-form with components

$$F_{\alpha}^{\nu} = \eta^{\nu\beta} F_{\beta\alpha} = \eta^{\nu\beta} (A_{\alpha,\beta} - A_{\beta,\alpha}) = A_{\alpha}^{;\nu} - A^{\nu}_{,\alpha} \tag{93}$$

i.e.,

$$\vec{F} = F = F_{\alpha}^{\nu} \frac{\partial}{\partial x^{\nu}} \otimes dx^{\alpha} = \overline{d\vec{A}} - d\vec{A} \tag{94}$$

It follows from equations (86), (93), and (94) that

$$\vec{F} \lrcorner \dot{\gamma} = (A_{\alpha}^{;\nu} \dot{x}^{\alpha} - A^{\nu}_{,\alpha} \dot{x}^{\alpha}) \frac{\partial}{\partial x^{\nu}} = \overline{d\vec{A}} \lrcorner \dot{\gamma} - d\vec{A} \lrcorner \dot{\gamma} \tag{95}$$

Using equations (87) and (91), we can rewrite equation (85) in the compact vector form

$$d_{\dot{\gamma}}\vec{\pi} = \varepsilon\overrightarrow{dA} \lrcorner \dot{\gamma} \tag{96}$$

All of the vectors in this equation are linear vectors. In the same vector notations, equation (75) can be rewritten as follows:

$$\dot{\gamma} = \vec{\pi} - \varepsilon\vec{A} \tag{97}$$

We propose now to interpret the spacetime geometric model of phase space momentum-energy as the affine momentum-energy of a classical charged particle [cf. equation (28) of Section 3], i.e.,

$$\hat{\pi} = \hat{0} \oplus \dot{\gamma} \tag{98}$$

More precisely,

$$\dot{\gamma} = \delta(\hat{\pi}, \hat{0}) \tag{99}$$

Comparison of equations (99) and (97) then yields

$$\delta(\hat{\pi}, \hat{0}) = \vec{\pi} - \varepsilon\vec{A} \tag{100}$$

which implies

$$d_{\dot{\gamma}}\delta(\hat{\pi}, \hat{0}) = d_{\dot{\gamma}}(\vec{\pi} - \varepsilon\vec{A}) \tag{101}$$

Also, from equations (13) and (25) it follows that

$$d_{\dot{\gamma}}\delta(\hat{\pi}, \hat{0}) = \hat{D}_{\dot{\gamma}}\hat{\pi} - \hat{D}_{\dot{\gamma}}\hat{0} = \hat{D}_{\dot{\gamma}}\hat{\pi} + \varepsilon\vec{F} \lrcorner \dot{\gamma} \tag{102}$$

Using the "semihamiltonian" equation (96) and relation (95), we can also write

$$d_{\dot{\gamma}}(\vec{\pi} - \varepsilon\vec{A}) = \varepsilon\overrightarrow{dA} \lrcorner \dot{\gamma} - \varepsilon d\vec{A} \lrcorner \dot{\gamma} = \varepsilon\vec{F} \lrcorner \dot{\gamma} \tag{103}$$

Substituting equations (102) and (103) in equation (101), we obtain

$$\hat{D}_{\dot{\gamma}}\hat{\pi} = 0 \tag{104}$$

Thus, the affine canonical momentum-energy of the classical charged particle is affine covariantly constant along the particle world line.

6. AFFINE REINTERPRETATION OF SUPERHAMILTONIAN DYNAMICS

In Sections 4 and 5 we showed that an affine structure is implicitly present in the superhamiltonian dynamics of classical charged particles in electromagnetic fields. We considered two equivalent superhamiltonian systems in Section 4. In system (I) the superhamiltonian (53) is "canonical," while the symplectic form (59) in noncanonical, and in system (II) the superhamiltonian is in the noncanonical form (72), while the symplectic form (52) is canonical. The equivalent of the two systems was shown to follow from the special coordinate transformation (70) on T^*M, which may be viewed as a translation in the local momentum–energy spaces.

We now want to investigate a third case that is trivial and at the same time informative. We need to reconsider the uncharged free-particle system where both the superhamiltonian and symplectic form are canonical, and then transform the system to noncanonical coordinates using the translation (70). This case is trivial in that the translations cannot change the integral curves of the system since $dH = -S \lrcorner X$ is an invariant equation. On the other hand, we can gain insight into the affine structure from the form of the translated Hamilton equations.

Denote by (III) the free-particle system with canonical superhamiltonian (53) and canonical symplectic form (52), and by (IV) the system obtained from (III) by applying translation (70). The essential results of the superhamiltonian formulations for these two systems are listed in the columns labeled (III) and (IV) in Table I. The last row shows the invariance of the spacetime equations of motion under the translations (70).

The entries in the third column reveal an important aspect of the affine structure. In particular, note that in the uncharged system (IV) the dynamical

Table I. Comparison of the Description of Free Uncharged Particles under Translation (Redefinition) of Reference Momentum-energy, in Superhamiltonian and Affine Formulations

	Superhamiltonian		Affine	
Description	III	IV	III	IV
Canonical momentum-energy	Coordinate π_μ in phase space		Affine vector in $(\hat{T}_pM, T_pM, \delta_0)$	
Reference momentum-energy	$\vec{0}$	$-q\vec{A}$	$\hat{z}$	$\hat{A} = \hat{z} \oplus -\varepsilon\vec{A}$
Superhamiltonian	$\pi^2/2m$	$(\pi - qA)^2/2m$	—	—
Difference function	—	—	δ_0	δ_0
Symplectic 2-form	$d\pi_\mu \wedge dx^\mu$	$d\pi_\mu \wedge dx^\mu + qF_{\mu\nu}\, dx^\mu \wedge dx^\nu$	—	—
R^4 connection	—	—	${}^{\hat{z}}K_\mu \hat{\lambda} = 0$	${}^{\hat{A}}K^\mu_\lambda = \varepsilon\nabla_\lambda A^\mu$
Hamilton's first equation	$(d\mathscr{H} + S \lrcorner X)(\partial/\partial\pi_\mu) = 0$		Spacetime curve $\gamma(s)$	
defines π in terms 4-velocity $\dot{\gamma}$	$\pi = m\dot{\gamma}$	$\pi = m\dot{\gamma} + qA$	$\hat{\pi} = \hat{z} \oplus \dot{\gamma}$	$\hat{\pi} = \hat{A} \oplus (\dot{\gamma} + \varepsilon A)$
Hamilton's second	$(d\mathscr{H} + S \lrcorner X)(\partial/\partial x^\mu) = 0$		$\hat{D}_{\dot{\gamma}}\hat{\pi} = 0$	
dynamical equation of motion	$\dot{\pi} = 0$	$\dot{\pi} - q\, dA/ds = 0$	$d^{\hat{z}}_{\dot{\gamma}}\pi = 0$	$d^{\hat{A}}_{\dot{\gamma}}\pi - \varepsilon\, dA/ds = 0$
Spacetime equation of motion	$d_{\dot{\gamma}}\dot{\gamma} = 0$		$d_{\dot{\gamma}}\dot{\gamma} = 0$	

force that generates nontrivial $\dot{\pi}_\mu$ is the term $q\, dA_\mu/ds$. This is in fact the term that gets transferred from the right to the left side of the dynamical equation in passing from (I) to (II). Systems (III) and (IV) reveal that this term is the result of simply redefining the reference of momentum-energy using the vector potential A_μ, and in the affine formulation may thus be considered geometrically as arising from an *integrable* R^4 connection.

To see this more precisely, consider a global inertial frame in Minkowski spacetime with electromagnetic field $F = dA$. We may take this global frame, and all frames obtained from $O(1, 3)$ rotations, as defining instantaneous comoving observers for all uncharged test particles at all points of spacetime, and take the zero vector field $\vec{0}$ as the reference of zero momentum-energy for uncharged test particles. The observer may choose to redefine the reference of momentum-energy using the vector potential by

$$\vec{0} \to \vec{0} - q\vec{A} \tag{105}$$

Since uncharged particles do not couple directly to the electromagnetic field, the momentum-energy of an uncharged particle is represented by a tangent vector in the affine theory. However, we may give each local tangent space its natural affine structure, denote the zero vector by $\hat{z}$, and assign an integrable R^4 connection by ${}^{\hat{z}}K^\mu_\lambda = 0$. Under the translation $\hat{z} \to \hat{z} \oplus -q\vec{A}$ this R^4 connection transforms to [cf. equation (10)]

$${}^{\hat{z}}K^\mu_\nu \to {}^{\hat{A}}K^\mu_\lambda = -qA^\mu_{,\lambda} \tag{106}$$

The last column in Table I shows that this term is responsible for the dA^μ/ds term in the dynamical Hamiltonian equation. This proves our claim made above that this term corresponds to an integrable R^4 connection.

It is now clear that in transforming from $\hat{0}$ to $\hat{A}$ in systems (I) and (II) we are also transforming away an integrable piece of the R^4 connection. The components of the R^4 connection relative to $\hat{0}$ are

$$\hat{D}_\mu \hat{0}^\lambda = {}^{\hat{0}}K^\lambda_\mu = -q\nabla^\lambda A_\mu + q\nabla_\mu A^\lambda \tag{107}$$

The second piece on the right-hand side can be transformed away by the translation (70), so that in general the state $\hat{A} = \hat{0} \oplus -q\vec{A}$ with

$$\hat{D}_\mu \hat{A}^\lambda = {}^{\hat{A}}K^\lambda_\mu = -q\nabla^\lambda A_\mu \tag{108}$$

no longer has additive integrable terms. However, when the electromagnetic field is covariant constant, ${}^{\hat{A}}K$ in (108) can be further transformed to zero by a translation, since this situation corresponds to the R^4 integrable fields mentioned earlier. It is not difficult to show, using (108) and (10), that this is the case.

We have the result that relative to the state $\hat{A} = \hat{0} \oplus -q\vec{A}$ the R^4 connection generically has no additive integrable terms. It thus represents a state of *minimal* R^4-anholonomity. Since the state $\hat{A}$ corresponds to the condition that the symplectic form is canonical, this suggests that a reference affine momentum-energy field be considered holonomic when the corresponding phase space coordinates are canonical.

Taking a clue from this correspondence, let us reconsider system (I). The superhamiltonian is especially simple, but the symplectic form is non-canonical. Note, however, that the symplectic form (59) can be rewritten in the form

$$\begin{aligned} S &= d\pi_\mu \wedge dx^\mu + qF_{\nu\mu}\, dx^\nu \wedge dx^\mu \\ &= (d\pi_\mu - qF_{\mu\nu}\, dx^\nu) \wedge dx^\mu \\ &= \mathscr{D}\pi_\mu \wedge dx^\mu \end{aligned} \tag{109}$$

where we have formally defined $\mathscr{D}\pi_\mu$ to be

$$\mathscr{D}\pi_\mu := d\pi_\mu - qF_{\mu\nu}\, dx^\nu \tag{110}$$

This expression may be considered as providing a vector bundle definition (Hermann, 1975) of a generalized affine connection with flat linear curvature. Since T^*M may be considered as an associated bundle of the affine frame bundle $A(M)$, a vector bundle affine connection on T^*M is equivalent to a generalized affine connection on $A(M)$ (Norris, 1988).

In the notation (110) the superhamiltonian system can be expressed as

$$H = \pi^\mu \pi_\mu / 2m, \qquad S = \mathscr{D}\pi_\mu \wedge dx^\mu \tag{111}$$

The symplectic form is *covariant* canonical.

Now suppose that $s \to \tau(s) = (x^\mu(x), \pi_\nu(s))$ is a curve in phase space with tangent vector $\dot{\tau}(s) = \dot{x}^\nu\, \partial/\partial x^\nu + \dot{\pi}_\nu\, \partial/\partial \pi_\nu$. Evaluating the one-forms $\mathscr{D}\pi_\mu$ at the tangent vector $\dot{\tau}(s)$, we find

$$\mathscr{D}\pi_\mu(\dot{\tau}) = \dot{\pi}_\mu - qF_{\mu\nu}\dot{x}^\nu \tag{112}$$

This will vanish if $\tau(s)$ is an integral curve of the superhamiltonian dynamical system. We are led once again to the affine geodesic equation.

The R^4 affine reinterpretation is summarized in Table II. We put into correspondence with Hamilton's first equation (the definition of canonical momentum-energy in terms of linear 4-velocity) the assumption that the momentum-energy of a classical charged particle is represented as an affine vector $\hat{\pi}$ that defines the linear 4-velocity when measured relative to the charged vacuum $\hat{0}$ using the difference function δ:

$$\dot{x}^\mu = \partial H / \partial \pi_\mu \leftrightarrow \dot{x}^\mu = \delta(\hat{\pi}, \hat{0}) \tag{114}$$

Table II. Comparison of the Superhamiltonian and Affine Descriptions of Charged Particle Dynamics

	Superhamiltonian		Affine	
Description	I	II	I	II
Canonical momentum-energy	Coordinate π_μ in phase space		Affine vector in $(\hat{\Pi}_p M, T_p M, \delta)$	
Reference momentum-energy	$\vec{0}$	$-q\vec{A}$	$\hat{0}$	$\hat{A} = \hat{0} \oplus -\varepsilon\vec{A}$
Superhamiltonian	$\pi^2/2m$	$(\pi - qA)^2/2m$	—	—
Difference function	—	—	δ	δ
Symplectic 2-form	$d\pi_\mu \wedge dx^\mu + qF_{\mu\nu}\, dx^\mu \wedge dx^\nu$	$d\pi_\mu \wedge dx^\mu$	—	—
R^4 connection	—	—	${}^{\hat{0}}K^\mu_\lambda = -\varepsilon F^\mu_\lambda$	${}^{\hat{A}}K^\mu_\lambda = -\varepsilon \nabla^\mu A_\lambda$
Hamilton's first equation	$(d\mathcal{H} + S \lrcorner X)(\partial/\partial\pi_\mu) = 0$		Spacetime curve $\gamma(s)$	
defines π in terms of 4-velocity $\dot{\gamma}$	$\pi = m\dot{\gamma}$	$\pi = m\dot{\gamma} + qA$	$\hat{\pi} = \hat{0} \oplus \dot{\gamma}$	$\hat{\pi} = \hat{A} \oplus (\dot{\gamma} + \varepsilon A)$
Hamilton's second	$(d\mathcal{H} + S \lrcorner X)(\partial/\partial x^\mu) = 0$		$\hat{D}_{\dot{\gamma}}\hat{\pi} = 0$	
dynamical equation of motion	$\dot{\pi} = qF \lrcorner \dot{\gamma}$	$\dot{\pi} = q\overrightarrow{dA} \lrcorner \dot{\gamma}$	$d^{\hat{0}}_{\dot{\gamma}}\pi + K \lrcorner \dot{\gamma} = 0$	$d^{\hat{A}}_{\dot{\gamma}}\pi + {}^{\hat{A}}K \lrcorner \dot{\gamma} = 0$
Spacetime equation of motion	$\ddot{x}^\mu = \varepsilon F^\mu_\lambda \dot{x}^\lambda$		$\ddot{x}^\mu = \varepsilon F^\mu_\lambda \dot{x}^\lambda$	

Table II illustrates how these expressions transform under momentum-energy translations.

Finally we replace the dynamical Hamilton equation with the affine geodesic equation:

$$\dot{\pi}_\mu = -\frac{\partial H}{\partial x^\mu}\, dx^\mu + C_{\mu\nu}\dot{x}^\nu \leftrightarrow \hat{D}_{\dot{x}}\hat{\pi} = 0 \tag{114}$$

7. CONCLUSIONS

In this paper we have sought to understand the relationship of the $R^4 \subset P(4)$ affine and the superhamiltonian descriptions of the dynamics of classical charged particles in electromagnetic fields. Although the two formulations appear to have distinct underlying geometric structures, we have found a correspondence between them so that one may be reinterpreted in terms of the other.

In the R^4 theory the basic structure consists of an R^4 affine connection on the affine frame bundle $A(M)$ of spacetime together with a geometric difference function on the local momentum-energy affine spaces. On the other hand, the structure in superhamiltonian dynamics is a symplectic 2-form and a superhamiltonian (function) on the phase space manifold T^*M. However, if T^*M is viewed as a vector bundle over M, then we have shown that, by using equation (110), we may extract from the symplectic form a definition of a vector bundle affine connection. Since T^*M may be considered as an associated bundle of $A(M)$, (110) also leads to a unique R^4 affine connection with flat linear curvature on $A(M)$. The first part of the correspondence is thus

$$\text{symplectic 2-form } S \leftrightarrow R^4 \text{ affine connection } \hat{K}$$

From a study of the structure of the Hamilton equations under momentum–energy translations in phase space we have found that the superhamiltonian plays the same role as does the geometric difference function, namely to define the momentum–energy of a particle in terms of its 4-velocity. Thus,

$$\text{superhamiltonian } \mathscr{H} \leftrightarrow \text{affine difference function } \delta$$

To set up this last correspondence, we found it necessary to make a "gauge correspondence" assumption based on a comparison of the properties of both theories under translations of momentum–energy. The translation from noncanonical to canonical coordinates that reduces the symplectic form $S = d\pi_\mu \wedge dx^\mu + qF_{\mu\nu}\, dx^\mu \wedge dx^\nu$ to canonical form also reduces the R^4 connection to a minimal R^4-anholonomic form. In view of the compatibility condition (13), it is natural to consider the holonomic form ${}^{\hat{A}}K^\mu_\nu = -q\nabla^\mu A_\nu$ of the R^4 connection as providing the dynamics of the difference function δ. This led us to consider the field $\hat{0}$ defined by instantaneously comoving uncharged inertial observers as R^4-anholonomic, and the translated field $\hat{A} = \hat{0} \oplus -q\vec{A}$ as R^4-holonomic. Thus, the gauge correspondence assumption is

$$\text{canonical coordinates} \leftrightarrow R^4 \text{ holonomic origin field}$$

Since the reference field $\hat{0}$ has its basis in the operational definition of the Lorentz force law in instantaneously comoving inertial frames, this assumption gives a degree of physical significance to noncanonical coordinates, considering them as R^4-anholonomic reference energy–momentum gauges. The remaining features of the correspondence are shown in Table II.

A remark is in order concerning the affine connection assumed in the $P(4)$ theory. To avoid the historical confusion over the term *affine connection*, a connection on the affine frame bundle $A(M)$ of a manifold is referred to as a *generalized affine connection* (Kobayashi and Nomizu, 1963). When the translational part of a generalized affine connection corresponds to the soldering 1-form on the linear frame bundle $L(M)$, the connection is referred to as an *affine connection.* In this case the R^4 curvature is the ordinary torsion of the associated linear connection, and this is the arena for the metric-affine theories (Hehl *et al.*, 1976). It has been shown (Norris *et al.*,

1980) that the translational part of a generalized affine connection ϕ can be decomposed on $L(M)$ with respect to the soldering 1-form θ as

$$\phi = \rho\theta + \tau$$

Here ρ is a function and τ corresponds to a trace-free type $(1,1)$ tensor field. The choices $\rho = 1$ and $\tau = 0$ reduce the generalized affine connection to an affine connection, and thus the τ component plays no role in the metric-affine theories. In the $P(4)$ theory $\rho = 0$ and the R^4 connection resides in the complementary τ component. Moreover, the linear connection in the $P(4)$ theory is a torsion-free Riemannian metric connection. What we have referred to as the R^4 curvature might more properly be called *generalized torsion*, but we shall not do so. The $P(4)$ theory is therefore distinct from the metric-affine theories.

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